Exercise 11

Find the closed form function for the following Taylor series:

$$f(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

Solution

Looking in Appendix C.2, which lists Taylor series expansions for trigonometric functions, we see that the function is

$$f(x) = \tan x.$$

We can show that it is by using long division.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{1}{7!}x^7 + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{1}{6!}x^6 + \cdots}$$

$$\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots \\ \hline
1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots \\
 & \underline{x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \frac{1}{720}x^7 + \cdots} \\
 & + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \frac{1}{840}x^7 - \cdots \\
 & \underline{+ \frac{1}{3}x^3 - \frac{1}{6}x^5 + \frac{1}{72}x^7 - \cdots} \\
 & + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \cdots \\
 & \underline{+ \frac{2}{15}x^5 - \frac{1}{15}x^7 + \cdots} \\
 & + \frac{17}{315}x^7 - \cdots \\
 & \underline{+ \frac{17}{315}x^7 - \cdots} \\
 & + \cdots
\end{array}$$